Solow-Swan model

Reminder from tutorial 1

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Some disclaimers about this reminder. *This recap is provided as an additional support to help you connect the equations covered in the course to graphical representation and economic interpretation. The course and the tutorials remain the only reference. This reminder is not necessary, and by all means not sufficient, to study Macroeconomics 1. Should you have any question or remark, please reach out to* `clement.montes@ensae.fr` *or* `nina.stizi@ensae.fr`

Objectives of the reminder

1. Recall the key properties of the production function.
2. Define the growth rate of a variable.
3. Explain the law of motion of capital per unit of efficient labor.
4. Define the steady state of an economy.
5. Explain why the growth rate of the economy is not determined by the saving rate.
6. Recall the golden rule of capital accumulation.
7. Represent the dynamic evolution of endogenous variables following a shock outside the steady state.

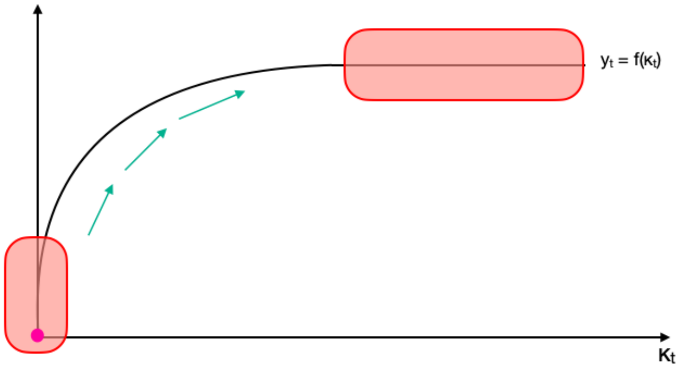
1 Properties of the production function

The economy produces output (Y), using capital (K) and efficient labor (AL). The production of the economy is such that $Y = F(K, AL)$. F is such that:

Property	Mathematical definition	Intuition
Essentiality condition	$F(K,0) = F(0,AL) = 0$	If one input is missing, production is not possible.
The marginal productivity in each factor is strictly increasing	$F_K > 0, \quad F_L > 0$	An increase in any factor increases output at the margin.
F is strictly concave in both inputs	$F_{KK} < 0, \quad F_{LL} < 0$	If **only one** input increases by ν, the production is multiplied by less than ν.
F has Constant Returns to Scale (CRS)	$\forall \lambda \in \mathbb{R}$, $F(\lambda K, \lambda AL) = \lambda F(K, AL)$	If **all** inputs increase by ν, the production is also increased by ν.
Inada condition at zero	$\lim\limits_{K \to 0^+} F_K(K, AL) = +\infty$ $\lim\limits_{L \to 0^+} F_L(K, AL) = +\infty$	When an input is scarce (input quantity goes to 0), a slight increase in its quantity leads to an infinite increase in the production
Inada condition at infinity	$\lim\limits_{K \to +\infty} F_K(K, AL) = 0^+$ $\lim\limits_{L \to +\infty} F_L(K, AL) = 0^+$	When an input is abundant, a slight increase in its quantity doesn't affect the quantity produced

These assumptions underpin all the graphical representations we will use throughout the semester. The essentiality condition is illustrated by the pink dot in figure 1 where the production function intersects the origin. Strictly positive and diminishing marginal productivity of each factor is reflected in the upward direction of the blue arrows and in their progressively flatter orientation as capital becomes more abundant. The Inada conditions are captured by the colored areas in figure 1: when capital is scarce, a small increase has a large effect on output (the peaked region), whereas when capital is abundant, additional capital has only a marginal impact (the flat region). An analogous shape applies to labor. Finally, constant returns to scale are best illustrated in three dimensions: when both inputs (K and L) increase proportionally, output increases in the same proportion, implying a linear production surface.

Figure 1: Production in Function of the Per-Efficient Labor Unit Capital



Why are those properties useful? Throughout the course and tutorials, the Inada properties and concavity ensure the convergence to a unique equilibrium. Constant returns to scale (homogeneity of degree one) allow us to eliminate scale effects and focus on a single endogenous variable on the production side: capital per unit of effective labor.
In practice: A common specification is the Cobb–Douglas production function: $Y = AK^{\alpha}L^{1-\alpha}$, which respects all the above condition. To empirically test the model, we have compared the estimated value of α with its value according to Kaldor's stylized facts.

- α is an exogenous parameter
- α represents the share of remuneration of capital across national income.
  - Proof: if the market of inputs are in pure and perfect competition, inputs are paid at their marginal productivity (MP), which is for example $MP_K = \frac{\partial Y}{\partial K} = \alpha \frac{Y}{K}$. The total price paid for capital in the aggregate economy is $MP_K \times K = \alpha Y$ Rewriting the quantity in term of national income (which is Y) provides with the result.

2 Growth rate

Definition: The growth rate of a variable X_t is the percentage change in X_t within an infinitesimal time period t and $t + dt$. Mathematically, the quantity $\frac{\dot{X}_t}{X_t}$ captures it. You can think of that quantity as $\frac{X_{t+dt}-X_t}{X_t} \times \frac{1}{dt}$ when dt is really small. It coincides with the derivative with respect to time of the *logarithm* of the variable. Intuitively, it captures the speed at which X_t grows.

How to compute a growth rate in Neo-Keynesian models? Consider a variable of the form $X_t = \prod_k x_{k,t}$. One is interested in the growth rate of X_t. By taking the logarithm of this variable, it follows: $\log(X_t) = \sum_k \log(x_{k,t})$. Deriving both sides with respect to time, it follows that $\frac{\dot{X}_t}{X_t} = \sum_k \frac{\dot{x}_{k,t}}{x_{k,t}}$ where the right-hand side is the growth rate of X_t and the left-hand side, the sum of the rate of variation of the $x_{k,t}$ in percent.

In Solow-Swan, we are mainly interested in $\kappa_t = \frac{K_t}{A_t L_t}$, the stock of capital per unit of efficient labor. It typically follows:

$$\frac{\dot{\kappa}_t}{\kappa_t} = \frac{\dot{K}_t}{K_t} - \underbrace{\frac{\dot{A}_t}{A_t}}_{=g} - \underbrace{\frac{\dot{L}_t}{L_t}}_{=n} \tag{1}$$

with g the growth rate of technological change and n the growth rate of population both assumed exogenous.

What is Growth? Growth corresponds to the growth rate of production **per capita** at the steady state. It is economically meaningful because higher production growth translates into higher consumption possibilities over time. Importantly, growth is **not about the level of production at the steady state, but the growth rate** of production per capita **at the steady state**. Confusing these two concepts would be akin to equating the value of a function at a point with its derivative at that point.

3 Key equations

3.1 Extensive form

$$K_{t+dt} = K_t + I_t d_t - \delta K_t d_t \implies \frac{K_{t+dt} - K_t}{dt} = I_t - \delta K_t \tag{2}$$

When $dt \to 0$, this equation yields the **law of motion of capital**:

$$\boxed{\dot{K_t} = I_t - \delta K_t}$$

Intuitively, the only way to increase the stock of capital is to invest. However, possessing capital is costly, because it depreciates. Macroeconomic models assume that this depreciation is proportional to the stock of capital.

The proportionality assumption made in Solow-Swan provides $I_t = sY_t$ and further simplifies this equation to $\dot{K_t} = sY_t - \delta K_t = sF(K_t, A_t L_t) - \delta K_t$.

3.2 Intensive form

If one multiplies by κ_t on both sides of Equation (1), it follows that $\dot{\kappa_t} = \frac{\dot{K_t}}{A_t L_t} - (g+n)\kappa_t$. Combining this equation with the law of motion of capital obtained in the Solow-Swan framework, one obtains the law of motion of capital per unit of efficient labor:

$$\boxed{\dot{\kappa_t} = sy_t - \delta\kappa_t - (g+n)\kappa_t} \tag{3}$$

The left-hand side of the equation represents the evolution across time of the stock of capital per unit of efficient labor. The right hand-side has three terms. The first term represents investment, expressed in intensive form. The second term represents the depreciation of the stock of capital per-effective labor. The third term represents the **dilution** of the stock of capital per unit of efficient labor.

4 Steady state

Definition: at the steady state, initial conditions are such that all variables are not zero and grow at a constant rate.

Why are we interested in the steady state? It corresponds to an equilibrium of the economy. To study the impact of a shock to the economy, we model it as a deviation to the steady state. The convergence back toward that steady state provides guidance on the variation of aggregate variables.

Does the steady state exist in real life? It is hard to measure. We simply assume we were at the steady state before the study of a shock.

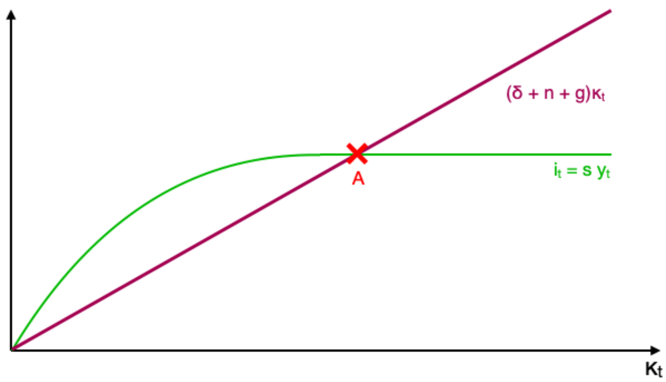
How to determine which initial conditions are needed? On the one hand, one must consider which variable is driving the endogenous dynamics. In the Solow-Swan model, the production can be fully re-expressed in terms of capital per-effective labor. Production is $y_t = f(\kappa_t)$ thanks to CRS. At the equilibrium of the good produced, aggregate production is aggregate demand, which guarantees $Y_t = C_t + I_t$. The consumption per-effective labor can thus be re-expressed in term of production per-effective labor $\gamma_t \equiv \frac{C_t}{A_t L_t} = (1-s)y_t$ thanks to the proportionality assumption. The dynamic of κ_t thus fully characterizes the dynamic of every variables of interest. On the other hand, initial conditions must correspond to **stock variables**, since flows can be adjusted by economic agents. In later models with endogenous consumption decisions, households will also optimally choose their consumption level at date 0. The only initial condition needed in Solow-Swan is κ_0.

Consequence at the steady state:

At the steady state $\frac{\dot{\kappa_t}}{\kappa_t} \overset{3}{=} s\frac{f(\kappa_t)}{\kappa_t} - \delta - n - g =$ constant. So $\frac{f(\kappa_t)}{\kappa_t}$ is constant across time, but f is concave and derivable, so $x \to \frac{f(x)}{x}$ is strictly increasing (similar to the derivative of f in 0), and surjective in $\mathbb{R}$, and thus bijective. It follows κ_t actually does not depend on time at the steady state. Eventually $\dot{\kappa_t} = 0$. **Remember that the definition of the steady state does not directly provide that all variables are constant across time: we demonstrated it for every framework.**

How to represent the steady state? As a consequence, at the steady state, the left hand side of equation (3) is 0: thus the force of accumulation of capital (i.e. investment) is perfectly counterbalanced by depreciation and dilution. This situation only happens at point A in figure 2.

Figure 2: Graphical Representation of the Steady State in the Solow-Swan Model



5 Determinant of growth

In the tutorial we derived the level of output per unit of efficient labor at the steady state. The production function included two types of capital: human and physical. Simplifying this expression to the case of a production function with only κ yields:

$$y^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}} \tag{4}$$

Equation 4 shows that the stedy-state level of production per-effective labor depends on the saving rate of the economy. However, again, the level of production at the steady state is different than the growth rate of the economy.

$$\frac{Y_t}{L_t} = A_t y^* \implies \frac{(\dot{Y_t/L_t})}{Y_t/L_t} = \underbrace{\frac{\dot{A}_t}{A_t}}_{=g} + \underbrace{\frac{\dot{y^*}}{y^*}}_{=0} = g \tag{5}$$

Intuitively, the study of growth focuses on the long run. While the saving rate determines how the economy converges to the steady state, our primary interest is the mechanism driving increases in output per capita once the economy has reached the steady state. Graphically, a change in the saving rate shifts in figure 2 the green curve up or down, moving the steady-state point (for example, from $A \to B$). However, we are concerned with the determinants that drive output per capita at the steady-state (i.e. once we reached B in that example).

6 Optimal saving rate

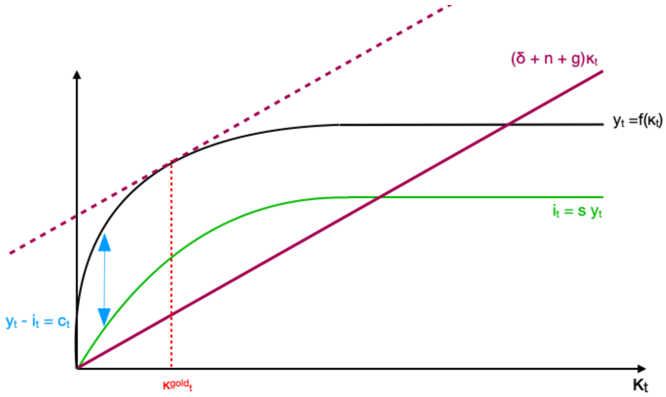
6.1 Golden rule of capital accumulation

The government is interested in the saving rate that maximizes consumption, which serves as a proxy for household welfare. Practically, even if the saving rate is exogenous, it makes sense to study such a scenario if you consider that aggregate savings are composed of public and private savings. A government could thus manipulate its saving rate by saving more or less, thus impacting the aggregate saving rate.
To determine s_{gold}, start the consumption per unit of efficient labor $c_t = (1-s)y^*$, replace the value of y^* -see equation (4)-, and differentiate with respect to s. Similarly, you can derive consumption with respect to the stock of capital per-effective labor, to find the steady state that maximizes consumption. This yields the golden rule of capital accumulation:

$$\boxed{f'(\kappa^*_{gold}) = \delta + n + g}$$

Graphically, the goal is to maximize consumption at the steady state. In figure 3, consumption corresponds to the vertical distance between the black production curve and the green investment curve. To find the steady state that maximizes consumption, we focus on the curves that remain fixed as the saving rate changes: the black production curve and the red

Figure 3: Graphical Representation on How to Find the Golden Rule



depreciation+dilution line. The golden rule states that, at the optimal steady state, the slope of the production function must equal the sum of depreciation and dilution. Graphically, this means finding the point on the black curve where a tangent can be drawn with the same slope as the red line. There is a unique $s_{\mathtt{gold}}$ allocating this stock of capital at the steady state.

6.2 Dynamic inefficiency

Carefully choosing the saving rate is important to avoid dynamic inefficiency, which occurs when savings are excessively high. Intuitively, a very high saving rate leads the economy to invest heavily in new capital. However, due to diminishing marginal productivity, this large capital stock generates only a small increase in output. As a result, capital may depreciate faster than it contributes to production, making the economy inefficient.

In other words, starting from such an inefficient point, it is possible to find a path along which consumption increases continuously until a new, steady-state with higher consumption. This gradual increase in consumption arises because households save less, reducing the capital stock. Graphically, this moves the economy from the right side of the horizontal colored area of the production function in figure 1 toward the left side of this colored area, where additional capital contributes more effectively to production.

7 Study of possible shocks

Since the government can identify the optimal saving rate, it can implement policies to guide the economy toward this level. Similarly, one can consider technological shocks that alter the production function, affecting the steady-state output and consumption.

If a shock happens, how does production react? Since, $y_t = f(\kappa_t)$ the response of output depends on both the shape of the production function and the stock of capital per unit of effective labor. The capital stock κ_t is a **stock variable, so it evolves continuously over time**[1].

As long as f is continuous, a shock to the economy **cannot** induce a discontinuous jump in output per unit of effective labor. Indeed, the capital stock k_t (or equivalently in per-effective-labor unit κ_t) cannot jump, hence the production cannot jump.

If a shock happen, how does consumption react? However consumption per unit of efficient labor is given by $c_t = (1 - s)f(\kappa_t)$. A sudden increase in the saving rate can make consumption jump discontinuously, since consumption is a flow variable[2]. If the shock instead affects the production function, both output y_t and consumption c_t can experience discontinuous changes, because $f(.)$ enters directly into both expressions.

[1] An exception is a sudden destruction of capital, such as from a natural disaster like an earthquake or a storm.

[2] Intuitively, consider the COVID-19 pandemic: when stores closed, you could not consume, but your capital stock was not destroyed. Consumption varied abruptly, while the stock of capital remained continuous.