# Cass–Koopmans–Ramsey model

*Reminder from tutorials 2 and 6*

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**Some disclaimers about this reminder.** *This recap is provided as an additional support to help you connect the equations covered in the course to graphical representation and economic interpretation. The course and the tutorials remain the only reference. This reminder is not necessary, and by all means not sufficient, to study Macroeconomics 1. Should you have any question or remark, please reach out to* clement.montes@ensae.fr *or* nina.stizi@ensae.fr

**Objectives of the reminder**

1. Recall the key properties of the utility.
2. Derive the instantaneous and intertemporal budget constraints.
3. Analyze Euler's equation.
4. Recall equilibrium conditions.
5. Represent the steady state and the dynamics of convergence to steady-state (phasis diagram).
6. Study of a shock (Tutorial 2 and 6).

## 1 Properties of the utility function

The Lucas critique spurred the microfounding of the consumer side. Solow-Swan's proportionality assumption left no scope for consumer adaptation to shocks. Cass-Koopman-Ramsey (CKR) addresses this by characterizing consumers, modeling the household's instantaneous utility with familiar assumptions. To do so they model the household's **instantaneous** utility thanks to assumptions that will sound familiar after having read the first recap.

| Property | Mathematical definition | Intuition |
|---|---|---|
| u is strictly increasing | $\frac{\partial u}{\partial c} > 0$ | The more consumption, the more utility |
| u is strictly concave | $\frac{\partial u}{\partial^2 c} < 0$ | Consumption brings fewer and fewer additional utility. Consumers prefer to smooth consumption over time. |
| Inada conditions | $\lim_{c \to 0} \frac{\partial u}{\partial c} = +\infty$ | When consumption is scarce ($c_t \to 0$), a slight increase in consumption induces an infinitely high additional utility ($u'(c_t) \to +\infty$). |
| | $\lim_{c \to +\infty} \frac{\partial u}{\partial c} = 0$ | While when consumption is abundant ($c_t \to +\infty$), a slight increase in consumption will not bring additional utility (i.e., marginal utility is null). |

The essentiality condition of the production function is no longer useful here since utility is ordinal and not cardinal. Utility will only depend on consumption in that framework, thus the CRS assumption would be equivalent to having linear utility. We cannot make it for the sake of generality.

Graphically: The assumptions yield a similar shape of the utility as the intuitions explained in the first recap for the production function.

Why are those properties useful? The Inada property is necessary for the existence of the steady state, see Part 1.2 of Problem Set 2.

The objective of the household, who is assumed infinitely lived, is her intertemporal utility:

$$\boxed{U_0 = \int_0^{+\infty} u(c_t)\exp(-\rho t)dt}$$

The exponential term, $\exp(-\rho t)$, is called discount factor: it allows to "express units from different time at the same period". We need to discount because throughout the model, the price of the good is assumed to be 1. We thus express everything in unit of goods. However, depending on the production at every date the good may be more or less scarce: this temporal variation needs to be accounted for. The discount factor is a function of the parameter of preference for present $\rho$.

# 2 Consumer's budget constraints

## 2.1 Instantaneous budget constraint

The representative household holds assets ($b_t$). If $b_t > 0$ the household can lend assets and receive interest for doing so. If $b_t < 0$ the household contracted a debt (i.e., borrowed) and has to pay interest to the lender. During a period of time $dt$, the household see the evolution of its stock of assets as being the sum of:

- **the financial wealth she previously held**, (+) $b_t$.
- **the remuneration from (or cost of) her financial wealth,** (+) $r_t b_t d_t$: if $b_t < 0$, she is indebted and pays an interest $r_t b_t$ at every instant $t$; conversely, if $b_t > 0$, she is a creditor and perceives an interest $r_t b_t$ at every instant. Hence, she pays (resp. perceives) $r_t b_t d_t$ during the period $dt$.
- **the revenue from her work,** (+) $w_t d_t$: since every household provides one unit of work at every instant, and the remuneration from labor is the wage rate $w_t$, she receives $w_t \times 1$ for work at every instant. Hence, she receives $w_t d_t$ from her work during the period $dt$.
- **the cost of her consumption** (−) $c_t d_t$: at every instant $t$, she consumes $c_t$, defined as the average consumption level in the population $c_t = \frac{C_t}{L_t}$. Hence she consumes $c_t dt$ during the period $dt$,

$$b_{t+dt} = b_t + \underbrace{r_t b_t dt}_{\text{financial income}} + \underbrace{w_t dt}_{\text{labor income}} - \underbrace{c_t dt}_{\text{consumption spending}}$$

Subtracting $b_t$, and dividing by $dt$ on both sides, then taking the limit of $dt \to 0$, and diving both sides by $L_t$ provides the **instantaneous budget constraint**.

$$\boxed{\dot{b_t} = r_t b_t + w_t - c_t} \tag{1}$$

## 2.2 Intertemporal budget constraint

However, the household wants to maximize her intermteporal utility, she just needs to compute her budget constraint on her "whole life". The idea is to integrate over all available periods equation (1) and find a way to reconstruct the path of assets since it does not make sense to have a $\dot{b_t}$ at the "end of her life".

1. Remark the integration by part. By substracting financial revenue on both sides, we obtain $\dot{b_t} - r_t b_t = w_t - c_t$. On the left hand side, it is possible to recognize the derivative of a product $u'v + uv'$ where $u = b_t$ and $v'$ encompasses $-r_t$. The idea is then to multiply both sides by an appropriate exponential term.
2. Time discount both sides by $\exp(-\int_0^t r_\tau d\tau)$. This term works perfectly for the integration by part. Intuitively, to connect this term to the discrete case you have already covered in your introductory classes to macroeconomics, one can remember those non-rigorous approximations: $\exp\left(-\int_0^T r_\tau d\tau\right) \overset{discretize}{\approx} \prod_{\tau=0}^t \exp(r_\tau)^{-1} \overset{r_\tau small}{\approx} \prod_{\tau=0}^t (1+r_\tau)^{-1} = \prod_{\tau=0}^t \frac{1}{1+r_\tau}$. You recognize the discount rate
3. Integrate both sides from 0 to $T$ an arbitrary date and solve for the left hand side.

It follows:

$$b_T \exp\left(-\int_0^T r_\tau d\tau\right) = b_0 + \int_0^T (w_t - c_t)\exp\left(-\int_0^T r_\tau d\tau\right) dt \tag{2}$$

The stock of assets at a given date depend on the initial endowment of the household $b_0$, its work revenue $(w_t)_t$ and consumption $(c_t)_t$ paths.

Now, take the limit when $T \to \infty$. The left hand side is the solvency constraint:

$$\boxed{b_T \exp\left(-\int_0^T r_\tau d\tau\right) \geq 0} \tag{3}$$

The right hand side yields the intertemporal budget constraint:

$$\boxed{\int_0^{+\infty} c_t \exp\left(-\int_0^T r_\tau d\tau\right) dt \leq b_0 + \int_0^{+\infty} w_t \exp\left(-\int_0^T r_\tau d\tau\right) dt} \tag{4}$$

The left hand side of equation (4) models the discounted consumption across all periods lived by the household. On the left hand side $b_0$ captures the initial endowment (i.e. if you are born reach, you may work less), and the second term represents the discounted labor revenues across time.

The solvency constraint allows to rule out Ponzi scheme

Remark that the intertemporal budget constraint is an inequality. It is binding at the optimum however, that is when the transversality condition applies.

# 3 Euler's equation

The program of the representative household is to maximize its **intertemporal** utility under constraints of :

- Positive consumption at every date
- Instantaneous budget constraint -equation (1)-
- Solvency constraint -equation (3)-

If one writes the Hamiltonian, it comes $H(c_t, b_t, \lambda_t, t) = \exp(-\rho t)u(c_t) + \lambda_t \left[r_t b_t + w_t - c_t\right]$

Two ways to derive the Euler equation

**Method 1 (always works)**

1. Derive the first order condition for the state variable: $\frac{\partial H}{\partial c_t} = 0$.
2. Remember the explicit expression of $\lambda_t$.
3. Differentiate the equation with respect to time.
4. Derive the FOC for the co-state variable: $\frac{\partial H}{\partial b_t} = -\dot{\lambda}_t$, and substitute $\lambda_t$ from step 2.
5. Merge the two first-order conditions.

**Method 2 (faster in simple cases)**

1. Derive the FOC for the co-state variable: $\frac{\partial H}{\partial b_t} = -\dot{\lambda}_t$. If it looks like $\lambda_t r_t = -\dot{\lambda}_t$, rewrite it as $\frac{\dot{\lambda}_t}{\lambda_t} = -r_t$.
2. Derive the FOC for the state variable: $\frac{\partial H}{\partial c_t} = 0$.
3. Re-express it in terms of $\lambda_t$, take logs, and differentiate with respect to time.
4. Merge the two equations.

Rearranging the two terms will yield:

$$\boxed{\frac{\dot{c}_t}{c_t} = \sigma(c_t)(r_t - \rho) \overset{CRRA}{=} \frac{r_t - \rho}{\theta}} \tag{5}$$

Methodology to interpret the Euler equation

1. Define the term and its economic interpretation.
2. State the variation of the growth rate of consumption per capita following a variation of the variable of interest, all other variables being equal.
3. **Give the economic mechanisms driving this variation.**

For example:

1. $r_t$ is the real interest rate. It measures the price of lending one unit of asset at period $t$.
2. If $r_t$ increases, all other variables being equal, the growth rate of consumption per capita increases

3. This is because, when the price of lending assets increases, it is advantageous for the household to lend assets, and thus to save rather than to consume. The household thus saves today, hence consumes less today. Tomorrow, she can consume more. Consequently, the growth rate of consumption increases.

$\rho$ is the preference for present parameter: the higher it is, the more households value present consumption. If $\rho$ increases, *ceteris paribus*, the growth rate of consumption decreases. Indeed, the more households value present, the more they will consume today (and thus the less they will save in assets). Tomorrow, they will have less resources to consume and will consume less.

$\theta$ is the risk aversion coefficient. Its inverse $\frac{1}{\theta}$ is the intertemporal elasticity of substitution (IES). The IES measures the extent to which households are willing to delay their consumption. The higher $\theta$, the less, all else being equal, households are willing to take risks on an uncertain future. Therefore, the less households are willing to reallocate their current consumption to a later date (that is why the higher the risk aversion, the lower the IES). The more they want to consume today instead of risking it to an uncertain future. They thus consume more today and have less resources to consume tomorrow: the growth rate of consumption decreases.

All in all, the behavior of the consumer is fully characterized by the Euler equation. Indeed, the two tradeoffs faced by the household **saving vs consuming at a given date**, or **intertemporal tradeoff of consumption today versus tomorrow** are fully expressed in this equation. To close the model, it is necessary to characterize the behavior of the other agent of the model: firms.

# 4 Equilibrium on markets

## 4.1 Firm side

The firm's production is completely determined by the economy's capital stock. Similar to Solow-Swan model (Chapter 1), firms operate under pure and perfect competition, precluding price optimization. Production is driven by aggregate demand for the final good, and firms optimize only on input usage. Labor and technology are exogenous, leaving the capital stock as the sole endogenous variable. As we saw in the Tutorial 1 recap: $\dot{\kappa_t} = i_t - (g + n - \delta)\kappa_t$

## 4.2 Clearing of the good-market

The economy's production is pinned down by demand, captured by the equilibrium on the final good via the standard **national accounting formula**. In a closed economy, the good-market clears when aggregate production $Y_t$ is equal to aggregate demand ; aggregate demand being the sum of aggregate consumption $C_t$, aggregate investment $I_t$, and public spending $G_t$. In most of the models seen this semester, we will assume $G_t = 0$. We will relax this assumption in Tutorial 6. If one injects this equilibrium condition in the law of motion of capital, it follows:

$$\dot{K}_t = Y_t - C_t - \delta K_t \implies \boxed{\dot{\kappa_t} = y_t - \gamma_t - (g + n - \delta)\kappa_t} \tag{6}$$

where $\gamma_t \equiv \frac{C_t}{A_t L_t}$ is the consumption per unit of efficient labor.

All in all, the behavior of households is fully characterized by consumption, the behavior of firms fully characterized by the stock of capital.

## 4.3 Indifference of households between the two financial markets

Households hold financial wealth which can be lent to firms or other households to increase their asset stock.

On the one hand (bond market), borrowing households demand funds to finance their consumption (= consumption credits) to crediting households in the form of a one-period risk-less bond. The bond market is at equilibrium when aggregate bond demand from borrowing households equals aggregate bond supply from crediting households, yielding the equilibrium price $r_t$, namely the real interest rate.

On the other hand (loans market), firms demand credit in the form of a loan to finance their capital. When the cost of borrowing is lower than the marginal cost of capital net of depreciation, $z_t - \delta$, firms keep demanding more loans. The loan market is at equilibrium when aggregate loan demand from firms equals aggregate loan supply from crediting households, yielding the equilibrium price on the loan market $z_t - \delta$.

With complete markets, the credit demands of firms and households are met by crediting households. Therefore, households are indifferent between providing credit as a one-period risk-free bond or as a loan to finance capital. In other words, lending on both markets is equivalent to crediting households. Consequently, bonds and loans on capital must yield the same return. At market equilibrium, these returns are $z_t - \delta$ and $r_t$, respectively. Indifference implies $r_t = z_t - \delta$.

Furthermore, under perfect competition in the goods market, firms are price-takers, setting prices equal to their marginal production cost. Thus, productive inputs are compensated according to their marginal product. Firm optimization yields $z_t = f'(\kappa_t)$. Consequently, the indifference condition becomes:

$$\boxed{r_t = f'(\kappa_t) - \delta} \tag{7}$$

Equation (7) links the behavior of the household and the firm in equilibrium. When interpreting Euler's equation in equation (5), avoid mentioning production (more a mechanism on the firm side). But when replacing the interest rate using equation (7), you implicitly discuss households' behavior at equilibrium and can interpret production as potentially impacting consumption.

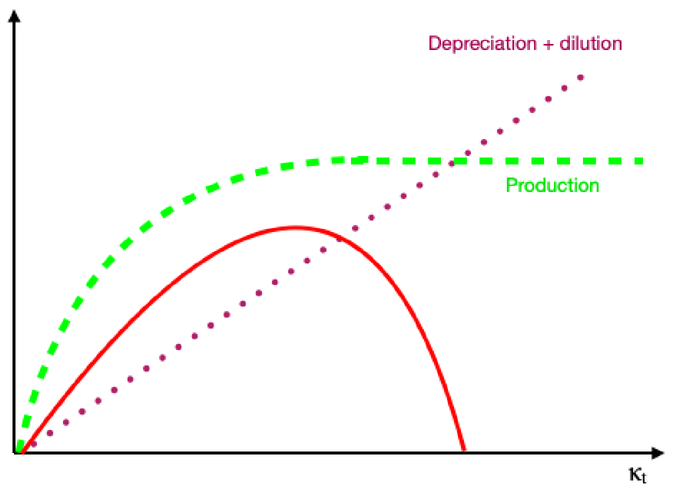
## 5 Phasis diagram

As usual, we are interested in the steady state of the economy. When defining it in CKR model, we demonstrated that all variables are actually constant in at the steady state. At the steady state:

$$\begin{cases} \gamma^* = f(\kappa^*) - (g + n + \delta)\,\kappa^* & \text{(L1)} \\ f'(\kappa^*) = \delta + \rho + \theta g & \text{(L2)} \end{cases}$$

Equation (L2) clearly defines a straight vertical line in the plan $(\kappa_t, \gamma_t)$. The shape of (L1) in that same plan is a bell curve.

Figure 1: Intuition Behind the Bell Curve Derived from the Law of Motion of Capital



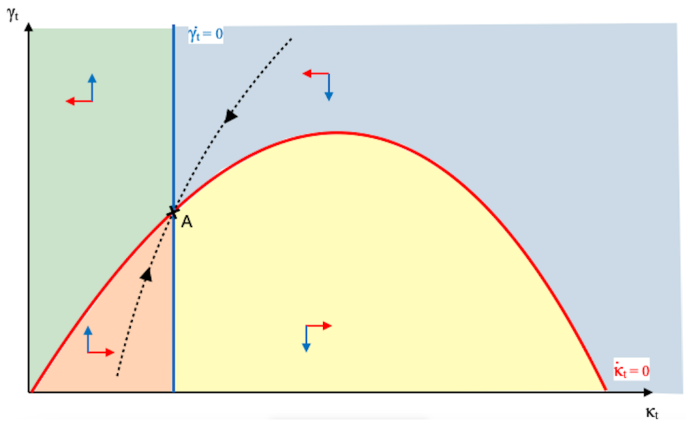

Notes: The green curve represents production per unit of efficient labor $y_t = f(\kappa_t)$. The purple 45 degree line represents the depreciation $\delta\kappa_t$ and dilution $(n + g)\kappa_t$.

On the right hand side of (L1) there are two terms. The first term is represented by the green dashed curve, the second by the purple dotted curve. The red bell curve represents the difference of the two functions. Intuitively, on the increasing part of the bell, the production sharply increases (because of Inada condition in 0), and thus dominates the linear impact of depreciation + dilution. The form of the red curve is thus very similar to the one of the green dashed production curve. At some point the decreasing marginal productivity will slow the increase of the green dashed production curve, and the difference with the purple dotted line will drag down more and more the green dashed curve. It follows the decreasing part of the bell curve. Note that the graphical illustration we do are purely **qualitative** and they aim at apprehending the way endogenous aggregates vary following a shock. The shape of the bell curve is thus not exact.

If one represents the two equations that need to be simultaneously satisfied on the same plan it follows:

When representing the two curves, it is important to draw the straight line at the left of the maximum of the bell curve. If one studies an economy where the curve $\dot{\gamma}_t = 0$ is at the right, it means that at the steady state, by using less capital, it would be possible to increase consumption at all point in time. The steady state would thus be unstable and we would be in the case of **dynamic inefficiency**. This case is ruled out by the assumption $\rho - n > (1 - \theta)g$

Figure 2: Phasis Diagram of the Convergence Towards Steady State in Cass-Koopman-Ramsey Model




Eventually, remember one goal of macroeconomic models: study the trajectory of the economy once a shock make the economy quit the steady state. It is then convenient to study how $\gamma_t$ and $\kappa_t$ are evolving in this diagram, as soon as we are outside the red and blue curves.

| **Variation of** | **Position in the plan** | **Movement** | **Intuition** |
|---|---|---|---|
| $\kappa_t$ | Above the bell curve i.e. $(\kappa_t\ ;\ \gamma_t)$ such that $\gamma_t > f(\kappa_t) - (g+n+\delta)\kappa_t$ | Horizontally to the left | Savings $(= f(\kappa_t) - \gamma_t)$ are lower than depreciation + dilution effect $(= (g+n+\delta)\kappa_t)$, so capital deteriorates faster than it is replaced, hence capital stock shrinks $(\dot{\kappa_t} < 0)$. |
| | Below the bell curve i.e. $(\kappa_t\ ;\ \gamma_t)$ such that $\gamma_t < f(\kappa_t) - (g+n+\delta)\kappa_t$ | Horizontally to the right | Savings are higher than depreciation + dilution effect, so capital accumulates faster than it deteriorates, hence capital stock increases. |
| | On the bell curve at point A. | Remain here unless a shock. | Savings exactly offset depreciation + dilution effect. Capital accumulates as fast as it degrades. So, the aggregate capital stock remains constant. Consumption level is also optimal, hence agents do not change their behavior. The economy henceforth remains at the steady-state (unless a shock / surprise occurs). |
| $\gamma_t$ | On the right of the vertical line i.e., $(\kappa_t\ ;\ \gamma_t)$ such that $f'(\kappa_t) < \delta + \rho + \theta g$ $\iff f'(\kappa_t) < f'(\kappa^*)$ $\iff \kappa_t > \kappa^*$ | Vertically downwards | Capital is abundant (because $\kappa_t > \kappa^*$). Because marginal returns are strictly decreasing (i.e., $f'$ is strictly decreasing), it yields lower returns that at the steady state (i.e., $f'(\kappa_t) < f'(\kappa^*)$), which negatively affect the returns from savings (which are equal to $f'(\kappa_t) - \delta$, that is the capital returns net of depreciation). Hence, households are saving less, so consuming more today than tomorrow, explaining the shrinkage of the consumption level over time (i.e., $\dot{\gamma_t} < 0$). |
| | On the left of the vertical line i.e., $(\kappa_t\ ;\ \gamma_t)$ such that $f'(\kappa_t) > \delta + \rho + \theta g$ $\iff \kappa_t < \kappa^*$ | Vertically upwards | Capital is scarce, so it yields high returns, which positively affect the returns from savings. Hence, households are savings more today to consume more tomorrow, explaining the rise of the consumption level over time. |
| | On the vertical line at point A. | Remains here unless a shock. | The economy is at its steady-state and stays there unless a shock occurs. In such case, consumption level will jump upwards (resp. downwards) along the vertical line on the impact of the positive (resp. negative) unanticipated surprise. |

What if the economy is on the bell curve but not at point A? Then, movement in $\gamma_t$, depending on the position of the economy on the plan (either left or right of the vertical line) will lead to future horizontal movements of $\kappa_t$.
Savings exactly offset depreciation + dilution effect. Capital accumulates as fast as it degrades. So, the aggregate capital stock remains constant ($\dot{\kappa_t} = 0$) but only for one instant $t$. If the consumption level at date $t$ is below (resp. above) its optimal point $\gamma^*$, then consumption increases (resp. decreases) between date $t$ and date $t + \delta t$. During $\delta t$, the consumption rises (resp. lessens), making savings insufficient (resp. more than enough) to compensate the capital degradation and so, at the next instant $t + \delta t$, capital stock will be lower (resp. higher) than at the previous instant $t$. So the economy will be on the green (resp. yellow) quadrant from the next instant.

What if the economy is on the vertical line but not at point A? Then, movement in $\kappa_t$, depending on the position of the economy on the plan (either up or down of the bell-shaped curve) will lead to future horizontal movements of $\gamma_t$.
Capital is exactly at its steady state value. If the consumption level at date $t$ is such that the economy at date $t$ described by $(\kappa^* \ ; \ \gamma_t)$ is above (resp. below) the bell-shaped curve, then the consumption level too high relative to $\kappa^*$ (resp. lower than what it could be considering $\kappa^*$), hence capital degrades faster (resp. slower) than it accumulates. This leads the capital stock at the next instant $t + \delta t$ to be lower (resp. higher) than at instant $t$. During the period $\delta t$, the economy moves horizontally to the left (resp. right) of the plan, putting the economy on the green (resp. yellow) quadrant, and upwards (resp. downwards) movement of future consumption are to expect.

Is it possible that the economy diverge following a shock? If the economy diverges, it can only do so by following the variation inside the yellow quadrant, or the green one. In the yellow quadrant, the economy increases its stock of capital but decreases its consumption. Asymptotically, $\gamma_t \to 0$ and $\kappa_t \to \bar{\kappa}$, with $\bar{\kappa}$ defined by the intersection of the bell curve to the x-axis. Thus production in the economy is high, but there is no consumption: the supply of good is in excess compared to the demand. Since the good market is cleared, production adjusts and the economy will converge to the equilibrium $(0, 0)$ which we ruled out in the definition of the steady state (mostly because it is not realistic). The same applies when $\gamma_t \to +\infty$ and $\kappa_t \to 0$ in the green quadrant. It is then straightforward that without production, consumption will fall to 0 and the economy would once again converge to $(0, 0)$.
All in all, even though it is mathematically possible to do so, it is economically meaningless and we will thus always study convergence back to a steady state.

# 6 Study of a shock in CKR model

**Methodology to graphically study the response to a shock**

1. Draw the temporal variation of the varying parameter/variable
2. Identify which curve is impacted by this variation
3. Identify the beginning and ending points of the trajectory of the economy
4. Determine the trajectory of the economy in the phasis diagram, in backward induction
5. **Interpret each movement in the plan thanks to economic mechanisms**

## 6.1 Unanticipated permanent shock on propensity to save (Tutorial 2)

1. Drawing the temporal variation of parameters/variables is important to understand which quadrants are active at which date. In figure 3b the blue quadrants are dictating the dynamic evolution of the economy before the shock. But as soon as the shock hits, at date $T$, the variables varies following the orange quadrants.

2. $\rho$ enters the equation (L2). A decrease of $\rho$ will increase of stock of capital at the new steady state.

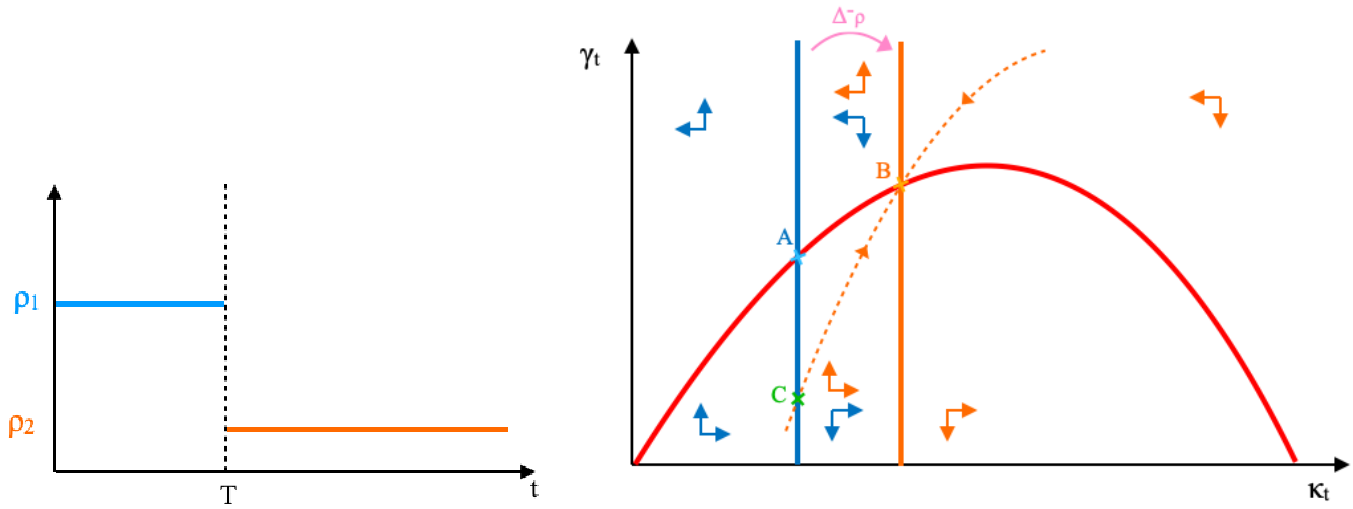
- *Mathematically*: since $f'$ is strictly decreasing, and thus $(f')^{-1}$, hence using the equation L2, we find that a permanent decrease in $\rho$ leads to an increase in $\kappa^*$.
- *Economically*: when households' preference for present declines (i.e., $\rho$ decreases) permanently, then it is less costly to sacrifice a unit today in order to make savings. This translates, all things equal, in an increase in the propensity to save ($r_t - \rho$ in the Euler equation, with $r_t = f'(\kappa_t) - \delta$ because of households' indifference between the two financial markets). So households save more permanently. Consequently capital accumulates.

Since $\kappa^*$ is in equation (L1), will the bell curve be impacted as well? No. Two ways of understanding it:

- *Mathematically*: the bell-shaped curve represents $\dot{\kappa_t} = 0$ that is $\gamma_t = f(\kappa_t) - (g + n + \delta)\kappa$ which does not involve $\rho$, so the curve's equation is unaffected by a change in $\rho$. The definition of the stock of capital per effective labour given by (L2) can be injected in (L1) only at the steady state. Even though $\kappa$ is affected by $\rho$, this $\kappa$ can enter (L2) only at the new steady state.
- *Economically*: The bell-shaped curve does not represent savings behavior, nor an equilibrium saving decision. Instead, it represents pure feasibility and a pure accounting identity. The locus $\kappa_t = 0$ represents the set of capital-consumption $(\kappa, \gamma)$ pairs for which net investment is zero given exogenous technology and demographics evolution. Put otherwise, it represent the feasible capital-consumption for which savings made by households (i.e., $f(\kappa_t) - \gamma_t$, equal to investments in capital in the case of national accounting), exactly offset the capital degradation proportional to the dilution and depreciation effect. As discussed above, a (permanent) decrease in the preference for the present will permanently increase (a) savings and (b) the stock of capital employed in the economy at steady-state. Put otherwise, a lower preference for the present alters households' intertemporal consumption choices, shifting the Euler equation (hence $\dot{\gamma_t}$) and the steady state (i.e., the pair $(\kappa^*, \gamma^*)$ ), but it does not modify the resource constraint governing capital accumulation. That is why the bell-shaped curve is unaffected by a change in households' rate of time preference because it reflects only the technological and demographic conditions under which capital remains constant.

3. As is often the case when studying a shock in macroeconomics, we assumed the economy is at the steady state before the shock. Since Figure 3a shows that before the shock, the economy was dictated by blue parameters, the economy starts in point A in Figure 3b. Once the shock has happened, the parameter describing the behavior of households is $\rho_2$. This parameter will now stay at that value and never come back to its initial value: the economy will thus reach a new steady state at point B in figure 3b.

Figure 3: Graphical Analysis of an Economy Following an Unanticipated Decrease in Preference for Present




(a) Preference for Present Across Time

(b) Dynamic of the Economy Following the Shock

4. In this precise example, there is just one date $T$. We know the economy will converge to $B$ in the long run, it thus must be on the saddle path toward that new steady state at the date of the shock. Additionally, **$\kappa$ is a stock and thus varies continuously[1]. It is thus not possible to have discontinuous horizontal (i.e. along the x-axis) jump.** Stated otherwise, it is not possible to jump from $A$ to $B$ in figure 3b. However, $\gamma$ is a flow and can vary discontinuously at the date of a shock. In that analysis, the trajectory in the diagram is thus $A \to C \to B$, where $A \to C$ is done along the blue line at date $T$, on impact of the decrease in $\rho$, and $C \to B$ will be done after date $T$ along the dashed orange saddle path.

5. When the economy is on the saddle path towards the steady state, the economy is at the equilibrium. It is thus possible to interpret mechanisms assimilating savings to investment.

- $A \to C$. At the date $T$, the household suddenly prefers consuming in the future, she delays consumption today to a later date by saving more in the present.

[1] Except in case of natural disaster.

- $C \to B$. This extra inflow of assets due to the increase in savings can be used by firms to invest in capital. Doing so, they accumulate capital faster that it deteriorates. These net extra inputs allow firms to produce more, and thus households to consume more in the long run.

## 6.2 Anticipated permanent shock on the final good market (Tutorial 6)

The framework seen in Chapter 2 relies on a simplified national accounting formula $Y = I + C$. In more general terms, public spending can easily be integrated in the framework by changing $Y = C + I + G$ with $G > 0$. While the program of the household is not impacted by this addition, the definition of investment in the law of motion of capital must be adjusted. The new system dictating the steady state is:

$$\begin{cases} \gamma^* = f(\kappa^*) - \chi^* - (g+n+\delta)\kappa^* & (L3) \\ \kappa^* = f'^{-1}(\delta + \rho + \theta g) & (L2) \end{cases}$$

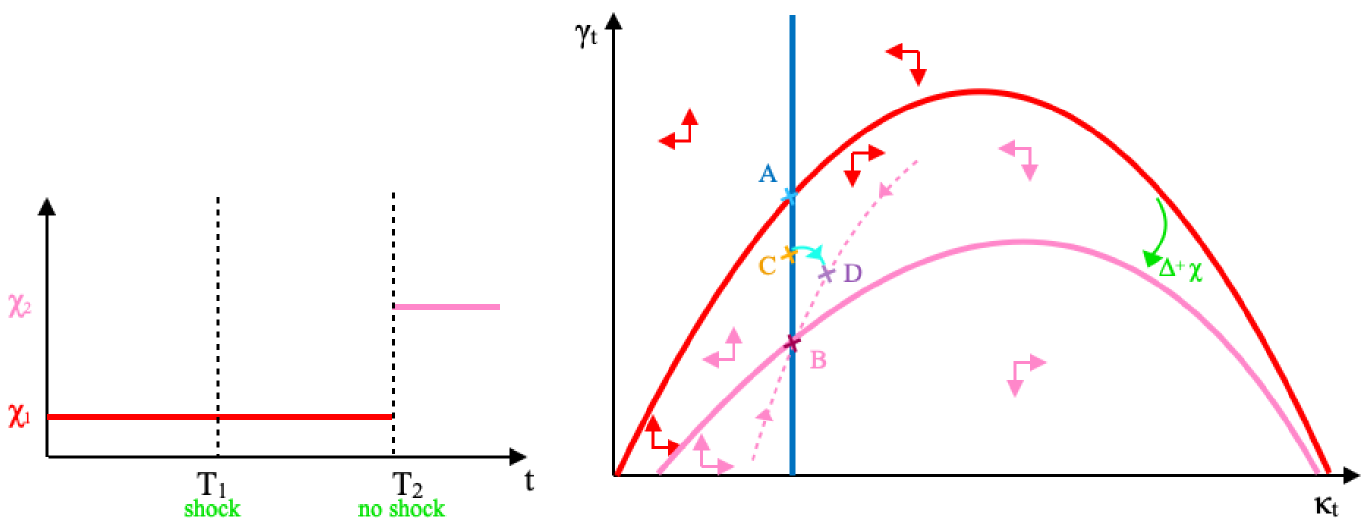
Intuitively, only the bell curve is impacted by the addition of this new term. It is shifted downward because the good produced needs to be shared by households to firms, and now the government. Actually, the public spending per unit of efficient labor may affect the shape of the bell, but again the diagrams are only meant to study qualitatively the variations of aggregates.

1. In figure 4b the red quadrants are dictating the dynamic evolution of the economy before date $T_2$ (even after the announcement at $T_1$). But as soon as the policy enters into force, at date $T_2$, the variables varies following the pink quadrants. The distinction between dates of shock/announcement and dates of entry into force are important because **consumption can vary discontinuously at date of shocks, however the household smooths her consumption otherwise**. It is a consequence of the concavity of her utility, and the fact that she optimizes on paths and not instantaneously. As a consequence, it is possible to have a jump in consumption in the diagram at date $T_1$, but not at date $T_2$.

2. $\chi$ enters the equation (L3). An increase of $\chi$ will decrease the consumption at the steady state since at a same aggregate demand, a larger share will be devoted to public spending.

3. We assumed the economy is at the steady state before the shock. With the same reasoning as before, the economy starts at coordinates $A$, and will end at coordinates $B$. Note that if the shock was temporary, and public spending went back to its initial level, the starting and ending points would be $A$.

Figure 4: Graphical Analysis of an Economy Following an Anticipated Increase of Public Spending




(a) Public Spending Across Time

(b) Dynamic of the Economy Following the Shock

4. At the date of the announcement $T_1$, the consumption is the only one to vary discontinuously from $A \to C$. By the absurd, if the consumption did not move, the economy would have staied in $A$ until $T_2$, where following the pink quadrant, it would have diverged North-West. By the absurd, if consumption jumped above $A$, or below $B$, it would diverge from $T_1$ on. By

the absurd, if consumption jumped at $T_1$ to $B$, between $T_1$ and $T_2$, following red quadrants, it would diverge South-East, and continue to diverge afterward. The only stable solution is thus to jump in-between $A$ and $B$. We then know that the only way for the economy to converge to $B$ in the long run is to be on the saddle path at date $T_2$, we then place point $D$. Eventually, between $T_1$ and $T_2$, the economy follows the red quadrant going South-East from $C \rightarrow D$, through the light-blue path.

5. Consumption jumps as a response to a surprise, here the announcement of a future change in a policy. Then, the economy follows the diagrams' dynamics: of the first diagram when $\chi = \chi_1$ ; and of the second diagram from the moment the level of public spending actually switches to $\chi = \chi_2$. This is because capital dynamics are still governed by the (effective) level of public spending and demographics and technological parameters. Anticipation do not enter in the capital dynamics hence do not affect the law capital accumulation until there is an actual change in either $\chi$, $\delta$, , or $f$. When the economy is on the saddle path towards the steady state of the second diagram, the economy is at the equilibrium.

- $A \rightarrow C$. At date $T_1$, the household learns she is intertemporally poorer because she will have to pay for this increase in public spending. She saves more in order to pay for later expenses.
- $C \rightarrow D$. Between $T_1$ and $T_2$, she keeps decreasing her consumption for the same reason, though in a continuous way, since she re-optimized intertemporally. The extra inflow of assets in the loan market allows firms to borrow more capital, and thus invest more, increasing their stock of capital.
- $D \rightarrow B$. From $T_2$ on, the stock of capital increased, thus the real interest rate will decrease in reaction. It follows that the growth rate of consumption decreases thanks to Euler equation. Indeed, since it is less profitable to save assets, the household consumes more today and hass less and less resources to consume in the future. The demand for the final good decreasing, the supply will adjust and thus the stock of capital also decreases.

Note that throughout the two subsections 6, we have studied a movement in both the straight line $\dot{\gamma}_t = 0$ and the bell curve $\dot{\kappa}_t = 0$. The mechanisms for any shock impacting CKR has thus been explored during the semester.